SECTION 14.1: VECTOR-VALUED FUNCTIONS

RECALL FROM COLLEGE ALGEBRA: A **function**, f, is a mapping which takes a real number input x and associates it with one and only one real number output, $f(x)$.

NOTE: Functions we've studied up to this point are often called '**real-valued**' functions since the outputs from such functions are **real numbers**.

DEFINITION: A **vector-valued function** ('vvf'), $\vec{r}$, is a mapping which takes a real number input t and associates it with only one **vector** output, $\vec{r}(t)$.

EXAMPLE 1: Lines such as $\vec{L}(t) = \langle 2-t, 3t, t+3\rangle$ are examples of vvf's.

RECALL: The **domain** of a function is the set of allowable **inputs**.

EXAMPLE 2: Let $\vec{r}(t) = \left\langle \ln(3-t), \dfrac{2}{t+1}, \sin^{-1}\left(\dfrac{t}{5}\right)\right\rangle$. Write the domain of $\vec{r}$ using interval notation.

- Domain of $\ln(3-t)$:

Ans: $(-\infty, 3)$

- Domain of $\dfrac{2}{t+1}$:

Ans: $(-\infty, -1) \cup (-1, \infty)$

- Domain of $\sin^{-1}\left(\dfrac{t}{5}\right)$:

Ans: $[-5, 5]$

- Domain of $\vec{r}$:

Ans: $[-5, -1) \cup (-1, 3)$

TAKEAWAY: The domain of vvf's go **component**-wise.

EXAMPLE 3: Let $\vec{r}(t) = \left\langle e^{-t}, t^2+2, \tan^{-1}(t) \right\rangle$. Find and simplify:

1. $\vec{r}(0)$ and $\vec{r}(-1)$

Ans: $\vec{r}(0) = \langle 1, 2, 0 \rangle$ and $\vec{r}(-1) = \left\langle e, 3, -\frac{\pi}{4} \right\rangle$

2. $\vec{r}(0) \cdot \vec{r}(-1)$

Ans: $\vec{r}(0) \cdot \vec{r}(-1) = e + 6$

3. $\vec{r}(0) \times \vec{r}(-1)$

Ans: $\vec{r}(0) \times \vec{r}(-1) = \left\langle -\frac{\pi}{2}, \frac{\pi}{4}, -2e+3 \right\rangle$

EXAMPLE 4: Let $\vec{r}(t) = \left\langle 1, 2t, 3t^2 \right\rangle$. Find and simplify:

1. $\vec{r}(t+h)$

Ans: $\vec{r}(t+h) = \left\langle 1, 2t+2h, 3t^2+6th+3h^2 \right\rangle$

2. $\vec{r}(t+h) - \vec{r}(t)$

Ans: $\vec{r}(t+h) - \vec{r}(t) = \left\langle 0, 2h, 6th+3h^2 \right\rangle$

3. $\dfrac{\vec{r}(t+h) - \vec{r}(t)}{h} = \dfrac{1}{h}\left[\vec{r}(t+h) - \vec{r}(t)\right]$

Ans: $\dfrac{\vec{r}(t+h) - \vec{r}(t)}{h} = \langle 0, 2, 6t+3h \rangle$

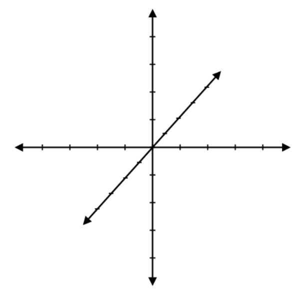
QUESTION: What does this last problem remind you of?

GRAPHS OF VECTOR-VALUED FUNCTIONS

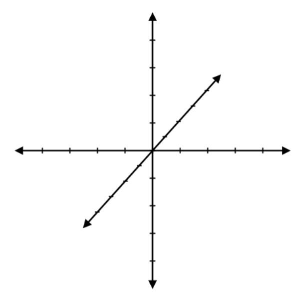
We use vector-valued functions to describe trajectories. To graph a vector-valued function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, we plot the vectors in standard position and visualize the function tracing out the terminal point of the vector along the curve described **parametrically** by the equations: $\{x = f(t), y = g(t), z = h(t)$.

EXAMPLE 5: Use a graphing utility to help you sketch or otherwise describe the following curves. Try to relate the components to each other algebraically to see if you can explain the shapes you're seeing.

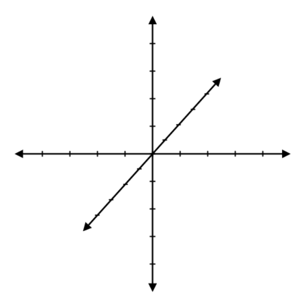
1. $\vec{r}(t) = \langle 3\cos(t), 3\sin(t), t \rangle$



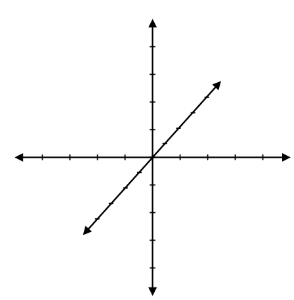
2. $\vec{r}(t) = \langle 3\cos(t), 3\sin(t), 3\cos(t) \rangle$



3. $\vec{r}(t) = \langle 3\cos(t), 3\sin(t), 3\cos(2t) \rangle$



4. $\vec{r}(t) = \langle 3\cos(t), 3\sin(t), 5e^{-t} \rangle$



EXAMPLE 6: Algebraically show the graph of $\vec{r}(t) = \langle 2\cos\left(e^{-t}\right), t, 2\sin\left(e^{-t}\right)\rangle$ lies on the cylinder $x^2+z^2=4$. Verify the claim graphically.

Ans: $x^2+z^2 = \left[2\cos\left(e^{-t}\right)\right]^2 + \left[2\sin\left(e^{-t}\right)\right]^2 = 4\left[\cos^2\left(e^{-t}\right)+\sin^2\left(e^{-t}\right)\right] = 4\,\checkmark$

EXAMPLE 7: Find a vector-valued function which traces out the curve of intersection of the two surfaces:

$$z = 4 - x^2 - y^2 \quad \text{and} \quad 2x + z = 4.$$

Check your answer graphically.

$\vec{r}(t) = \langle \cos(t)+1, \sin(t), -2\cos(t)+2\rangle$

LIMITS OF VECTOR-VALUED FUNCTIONS

RECALL: $\lim\limits_{x\to a} f(x) = L$ means given $\epsilon > 0$ there is a $\delta > 0$ so that if $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$.

QUESTION: Can you still draw the picture that goes with this definition and explain what it means?

We wish to extend the notion of limit to vector-valued functions! To that end, we 'vectorize' the above definition:

DEFINITION: $\lim\limits_{t\to a} \vec{r}(t) = \vec{L}$ means given $\epsilon > 0$ there is a $\delta > 0$ so that if $0 < |t-a| < \delta$, then $\|\vec{r}(t) - \vec{L}\| < \epsilon$.

QUESTION: Can you think of a picture that goes with this definition?

THEOREM: If $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ and $\vec{L} = \langle L_1, L_2, L_3\rangle$ then:

$$\lim_{t\to a} \vec{r}(t) = \vec{L} \iff \lim_{t\to a} x(t) = L_1, \lim_{t\to a} y(t) = L_2, \text{ and } \lim_{t\to a} z(t) = L_3$$

That is, **limits of vector-valued functions go component-wise**.

NOTE: Since the limits of vector-valued functions go component-wise we get all of our limit properties - including all-time favorites such as the substitution rule and L'Hopital's Rule.

EXAMPLE 8: Find the limits of the following vector-valued functions if they exist!

1. $\lim\limits_{t\to 2} \left\langle 3t-1, t^2, \tan^{-1}(1-t)\right\rangle$

Ans: $\left\langle 5, 4, -\dfrac{\pi}{4}\right\rangle$

2. $\lim\limits_{t\to 0} \left\langle \dfrac{\cos(4t)-1}{\tan(3t)}, \dfrac{\sin(3t)}{e^{2t}-1}, t^2\ln|t|\right\rangle$

Ans: $\left\langle 0, \dfrac{3}{2}, 0\right\rangle$

CONTINUITY OF VECTOR-VALUED FUNCTIONS

RECALL: f is **continuous** at $x = a$ if $\lim_{x\to a} f(x) = f(a)$. That is, f is continuous at $x = a$ if:

- $f(a)$ exists.
- $\lim_{x\to a} f(x)$ exists.
- $\lim_{x\to a} f(x) = f(a)$.

EXAMPLE 9: 'Vectorize' the above definition: a vector-valued function $\vec{r}$ is continuous at $t = a$ means:

NOTE: Since limits of vector-valued functions go component-wise, so does continuity. Hence...

RULE OF THUMB: Most of the time... vector-valued functions are continuous on their domains.

EXAMPLE 10: List the intervals of continuity of the following vector-valued functions:

1. $\vec{r}(t) = \left\langle 3t^2 - 2t + 1, \dfrac{\cos(3t)}{t^2+1}, \tan^{-1}(5t) \right\rangle$

Ans: $(-\infty, \infty)$

2. $\vec{r}(t) = \left\langle \ln(3-t), \dfrac{2}{t+1}, \sin^{-1}\left(\dfrac{t}{5}\right) \right\rangle$

Ans: $[-5, -1)$, $(-1, 3)$

HOMEWORK: Section 14.1: 9 - 55 odd